

On the reflexivity of $\mathcal{P}_w(^n E; F)$

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Abstract

In this paper we prove that if E and F are reflexive Banach spaces and G is a closed linear subspace of the space $\mathcal{P}_w(^n E; F)$ of all n -homogeneous polynomials from E to F which are weakly continuous on bounded sets, then G is either reflexive or non-isomorphic to a dual space. This result generalizes [5, Theorem 2] and gives the solution to a problem posed by Feder [4, Problem 1].

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1 Introduction

An important result of Feder [4] states that if E and F are reflexive Banach spaces such that F or E' is a subspace of a Banach space with an unconditional basis, then the space $\mathcal{L}_K(E; F)$ of all compact linear operators from E to F is either reflexive or non-isomorphic to a dual space. In [5], Feder and Saphar proved that if E and F are reflexive Banach spaces and G is a closed linear subspace of $\mathcal{L}_K(E; F)$ which contains the space $\mathcal{R}(E, F)$ of all finite rank linear operators from E to F , then G is either reflexive or non-isomorphic to a dual space. But the following question posed in [4] remains open:

Question. *Let E and F be reflexive Banach spaces. Is $\mathcal{L}_K(E; F)$ either reflexive or non-isomorphic to a dual space?*

In this paper, we obtain a positive answer for the previous question. In fact, we prove the following more general result:

Theorem. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{P}_w(^n E; F)$. Then G is either reflexive or non-isomorphic to a dual space.*

The answer of the aforementioned question is obtained just considering $n = 1$ in the previous theorem. As other consequences of this result we also obtain two conditions, one that ensures that $\mathcal{P}_w(^n E; F)$ is non-isomorphic to a dual space and other such that $\mathcal{P}_w(^n E; E)$ is non-isomorphic to a dual space (see Corollaries 2.8 and 2.9). We also obtain a generalization of Boyd and Ryan [2, Theorem 21].

Throughout this paper E and F denote Banach spaces over \mathbb{K} , where \mathbb{K} is \mathbb{R} or \mathbb{C} , E' denotes the dual of E and $B_E = \{x \in E : \|x\| \leq 1\}$. We say that E is a *dual space* if there exists a Banach space X such that $X' = E$. Let $J_E : E \rightarrow E''$ denotes the canonical injection from E into E'' . The space of all bounded linear operators from E to F is represented by $\mathcal{L}(E; F)$ and $\mathcal{P}(^n E; F)$ denotes the Banach space of all continuous n -homogeneous

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polynomials from E into F with its usual sup norm. We omit F when $F = \mathbb{K}$. Let $\mathcal{P}_f(^n E; F)$ denotes the subspace of $\mathcal{P}(^n E; F)$ generated by all polynomials of the form $P(x) = (\phi(x))^n b$, with $\phi \in E'$ and $b \in F$. We denote by $\mathcal{P}_A(^n E; F)$ the closure of $\mathcal{P}_f(^n E; F)$ with the norm topology and $\mathcal{P}_w(^n E; F)$ denotes the subspace of $\mathcal{P}(^n E; F)$ formed by all P which are *weakly continuous on bounded sets*, that is the restriction $P|_B : B \rightarrow F$ is continuous for each bounded set $B \subset E$, when B and F are endowed with the weak topology and the norm topology, respectively. The subspace $\mathcal{P}_K(^n E; F)$ of $\mathcal{P}(^n E; F)$ is formed by all polynomials that send bounded sets onto relatively compact sets. It is well-known that

$$\mathcal{P}_w(^n E; F) \subset \mathcal{P}_K(^n E; F) \subset \mathcal{P}(^n E; F)$$

and $\mathcal{P}_w(^n E; F) = \mathcal{L}_K(E; F)$ when $n = 1$. For $T \in \mathcal{L}(E; F)$ we denote by $T' \in \mathcal{L}(F'; E')$ the adjoint operator of T . Finally, let us recall that E has the *compact approximation property* (CAP in short) if given a compact set $C \subset E$ and $\epsilon > 0$, there is $T \in \mathcal{L}_K(E; E)$ such that $\|Tx - x\| < \epsilon$ for every $x \in C$.

2 The main result

To prove the main result, we need the following lemma, which is a special case of [6, Corollary 5].

Lemma 2.1. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{P}_w(^n E; F)$. Let $P_m, P \in G$ for each $m \in \mathbb{N}$. Then $\lim_{m \rightarrow \infty} P_m = P$ weakly in G if and only if $\lim_{m \rightarrow \infty} y'(P_m(x)) = y'(P(x))$ for every $x \in E$ and every $y' \in F'$.*

Theorem 2.2. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{P}_w(^n E; F)$. Then G is either reflexive or non-isomorphic to a dual space.*

Proof. Suppose that G is isomorphic to the conjugate of a Banach space X . Let $\varphi : X' \rightarrow G$ be an isomorphism. To show that G is reflexive, we need to prove that every sequence in B_G has a weakly convergent subsequence. Consider (P_m) in B_G . Since $B_{G''}$ is $\sigma(G'', G')$ -compact there exist a subsequence $(J_G(P_{m_k}))$ of $(J_G(P_m))$ and $\theta \in G''$ such that $\lim_{k \rightarrow \infty} J_G(P_{m_k}) = \theta$ in the $\sigma(G'', G')$ -topology. For every $y' \in F'$ and $x \in E$, consider the linear functional

$$\psi_{y',x} : P \in G \rightarrow y'(P(x)) \in \mathbb{K}.$$

Since $\psi_{y',x} \in G'$ we have that

$$\lim_{k \rightarrow \infty} \langle J_G(P_{m_k}), \psi_{y',x} \rangle = \lim_{k \rightarrow \infty} y'(P_{m_k}(x)) = \theta(\psi_{y',x})$$

for every $y' \in F'$ and $x \in E$. We want to prove that

$$\pi : \phi \in G'' \rightarrow J_G \circ \varphi \circ J'_X \circ (\varphi'')^{-1}(\phi) \in J_G(G)$$

is a projection. Note that

$$\begin{aligned} \langle J'_X \circ (\varphi'')^{-1}(J_G(P)), z \rangle &= \langle (\varphi'')^{-1}(J_G(P)), J_X(z) \rangle = \langle (\varphi^{-1})''(J_G(P)), J_X(z) \rangle = \langle J_G(P), (\varphi^{-1})'(J_X(z)) \rangle \\ &= \langle (\varphi^{-1})'(J_X(z)), P \rangle = \langle J_X(z), \varphi^{-1}(P) \rangle = \langle \varphi^{-1}(P), z \rangle \end{aligned}$$

for each $P \in G$ and $z \in X$. This implies that

$$J'_X \circ (\varphi'')^{-1}(J_G(P)) = \varphi^{-1}(P)$$

and then $\pi \circ J_G = J_G$. Thus π is a projection and so

$$G'' = J_G(G) \oplus \ker(\pi).$$

Let $Q \in G$ and $\eta \in \ker(\pi)$ such that $\theta = J_G(Q) + \eta$. Since $\eta \in \ker(\pi)$ and $J_G \circ \varphi$ is injective, we have $J'_X \circ (\varphi'')^{-1}(\eta) = 0$. On the other hand,

$$\eta(\psi_{y',x}) = \langle (\varphi'')^{-1}(\eta), \varphi'(\psi_{y',x}) \rangle = \langle J''_X(\varphi'(\psi_{y',x}), (\varphi'')^{-1}(\eta)) \rangle = \langle \varphi'(\psi_{y',x}), J'_X \circ (\varphi'')^{-1}(\eta) \rangle = 0.$$

Hence

$$\lim_{k \rightarrow \infty} y'(P_{m_k}(x)) = \theta(\psi_{y',x}) = \langle J_G(Q), \psi_{y',x} \rangle + \langle \eta, \psi_{y',x} \rangle = y'(Q(x)),$$

for every $y' \in F'$ and $x \in E$. By Lemma 2.1 it follows that $\lim_{k \rightarrow \infty} P_{m_k} = Q$ weakly in G . This completes the proof. \square

The next result is just Theorem 2.2 with $n = 1$. This is also the affirmative answer of [4, Problem 1] and consequently a generalization of [5, Theorem 2] and [4, Theorem 5].

Corollary 2.3. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{L}_K(E; F)$. Then G is reflexive or non-isomorphic to a dual space.*

Remark 2.4. *Note that Theorem 2.2 does not work for $\mathcal{P}_K(^n E; F)$ instead of $\mathcal{P}_w(^n E; F)$. In fact, $\mathcal{P}_K(^2 \ell_2) = \mathcal{P}(^2 \ell_2) = \mathcal{L}(\ell_2; \ell_2)$ is a dual space that is not reflexive.*

The next result is a generalization of [2, Theorem 21].

Corollary 2.5. *Let E be a reflexive Banach space. Then $\mathcal{P}_A(^n E)$ is either reflexive or non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Corollary 2.6. *Let E and F be reflexive Banach spaces and G be a closed linear subspace of $\mathcal{P}_w(^n E; F)$. If $\mathcal{P}(^n E; F)$ is isomorphic to G , then $\mathcal{P}(^n E; F)$ is reflexive.*

Proof. Since $\mathcal{P}(^n E; F)$ is a dual space, then the conclusion follows from Theorem 2.2. \square

The next proposition is a particular case of [1, Proposition 5.3].

Proposition 2.7. *Let E and F be Banach spaces. Then $\mathcal{P}_w(^n E; F)$ is isomorphic to a closed subspace of $\mathcal{P}_w(^m E; F)$ for every $m \geq n$.*

Proof. To prove the proposition by induction on n it suffices to prove that $\mathcal{P}_w(^n E; F)$ is isomorphic to a closed subspace of $\mathcal{P}(^{n+1} E; F)$. Choose $\varphi \in E'$ such that $\varphi \neq 0$. Define $\rho : \mathcal{P}_w(^n E; F) \rightarrow \mathcal{P}_w(^{n+1} E; F)$ by $\rho(Q)(x) = \varphi(x)Q(x)$ for all $x \in E$. It is clear that ρ is an injective linear operator. Therefore $\mathcal{P}_w(^n E; F)$ is isomorphic to $\rho(\mathcal{P}_w(^n E; F)) \subset \mathcal{P}_w(^{n+1} E; F)$. This completes the proof. \square

Finally we obtain the following results.

Corollary 2.8. *Let E and F be reflexive Banach spaces such that E has the CAP. If $\mathcal{P}_w(^n E; F) \neq \mathcal{P}(^n E; F)$, then $\mathcal{P}_w(^m E; F)$ is not isomorphic to a dual space for every $m \geq n$.*

Proof. By Theorem 2.2 we only need to prove that $\mathcal{P}_w(^m E; F)$ is not reflexive for every $m \geq n$. By Proposition 2.7 we have that $\mathcal{P}_w(^n E; F)$ is isomorphic to a closed subspace of $\mathcal{P}_w(^m E; F)$ for every $m \geq n$. If we assume that $\mathcal{P}_w(^m E; F)$ is reflexive for some $m \geq n$, then $\mathcal{P}_w(^n E; F)$ is also reflexive. By [3, Corollary 4.4] we have that $\mathcal{P}_w(^n E; F) = \mathcal{P}(^n E; F)$, but this contradicts the hypothesis. \square

Corollary 2.9. *Let E be a reflexive infinite dimensional Banach space with the CAP. Then $\mathcal{P}_w(^n E; E)$ is non-isomorphic to a dual space for every $n \in \mathbb{N}$.*

Proof. By the Riesz Theorem $\mathcal{L}_K(E; E) \neq \mathcal{L}(E; E)$. Now the result follows from Corollary 2.8. \square

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